

ANALYSIS OF TIME-FREQUENCY SCATTERING TRANSFORMS

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ABSTRACT. In this paper, we address the problem of constructing a feature extractor which combines Mallat’s scattering transform framework with time-frequency (Gabor) representations. To do this, we introduce a class of frames, called *uniform covering frames*, which includes a variety of semi-discrete Gabor systems. Incorporating a uniform covering frame with a neural network structure yields the *Fourier scattering transform* $\mathcal{S}_{\mathcal{F}}$. We prove that $\mathcal{S}_{\mathcal{F}}$ propagates energy along frequency decreasing paths and its energy decays exponentially as a function of the depth. These quantitative estimates are fundamental in showing that $\mathcal{S}_{\mathcal{F}}$ satisfies the typical scattering transform properties, and in controlling the information loss due to width and depth truncation. We introduce the *truncated Fourier scattering transform* and the *fast Fourier scattering transform* algorithm, and illustrate the algorithm’s performance. The time-frequency covering techniques developed in this paper are flexible and give insight into the analysis of scattering transforms.

1. INTRODUCTION

Introduced by LeCun [LBD⁺90], a *convolutional neural network* is a composition of a finite number of transformations, where each transformation is one of three types: a convolution against a filter bank, a non-linearity, and an averaging. Convolutional neural networks approximate functions through an adaptive and iterative learning process and have been extremely successful for classifying data [LBD⁺90, HDY⁺12, KSH12]. Since they have complex architectures and their parameters are learned through “black-box” optimization schemes, training is computationally expensive and there is no widely accepted rigorous theory that explains their remarkable success.

Recently, Mallat [Mal12] provided an intriguing example of a predetermined convolutional neural network with formal mathematical guarantees. His *windowed scattering transform* $\mathcal{S}_{\mathcal{W}}$ propagates the input information through multiple iterations of the wavelet transform and the complex modulus, and finishes the process with a local averaging. It is typically used as a *feature extractor*, which is a transformation that organizes the input data into a particular form, while simultaneously discarding irrelevant information. When combined with standard classifiers, the windowed scattering transform has achieved state-of-the-art results for several classification problems [BM13, SM13, HPM15]. Mallat and collaborators showed that, by combining the wavelet transform with a neural network structure, it is possible to obtain a significantly more powerful feature extractor than just the wavelet transform itself.

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While Mallat’s results are impressive, there are several reasons to consider an alternative case where a non-wavelet frame is used for scattering.

- (a) Since neural networks are inspired by the structure of the brain, it makes sense to mimic the visual system of mammals when designing a feature extractor for image classification. The ground-breaking work of Daugman [Dau85] demonstrated that simple cells in the mammalian visual cortex are modeled by modulations of a fixed 2-dimensional Gaussian. In other words, this is a Gabor system with a Gaussian window.
- (b) The authors of [LEN08] observed that the learned filters (the experimentally “optimal” filters) in Hinton’s algorithm for learning *deep belief networks* [HOT06] are localized, oriented, band-pass filters, which resemble Gabor functions. Strictly speaking, this set of functions is not a Gabor system since it is not derived from a single generating function, but it is not a wavelet system either.
- (c) The use of Gabor frames for classification is not unprecedented, since the short-time Fourier transform with Gaussian window has been used as a feature extractor for various image classification problems [HUW⁺98, KZL03, AGP06]. These papers predated Mallat’s work on scattering transforms and did not use Gabor functions in a multi-layer decomposition. To the best of our knowledge, we are not aware of any prior work that combines neural networks with Gabor functions.

We address the situation where a Gabor frame is used for scattering. In Section 2, we introduce a new class of frames, called *uniform covering frames*, and this class is a natural generalization of certain types of Gabor frames. In fact, no wavelet frame is a uniform covering frame, so our situation is completely different from that in Mallat [Mal12]. We combine uniform covering frames with neural networks to obtain the *Fourier scattering transform* $\mathcal{S}_{\mathcal{F}}$.

In Section 3, we concentrate on the theoretical analysis of $\mathcal{S}_{\mathcal{F}}$. Since both $\mathcal{S}_{\mathcal{F}}$ and $\mathcal{S}_{\mathcal{W}}$ share the same network structure, it is natural to ask whether they share the same broad mathematical properties. The first main contribution of our paper is Theorem 3.6, which shows that the answer to this question is ‘yes’. Its proof heavily relies on Proposition 3.3, which establishes an exponential decay of energy estimate for $\mathcal{S}_{\mathcal{F}}$. It is unknown if $\mathcal{S}_{\mathcal{W}}$ satisfies this property.

In Section 4, we define the *truncated Fourier scattering transform* $\mathcal{S}_{\mathcal{F}}[M, K]$, where the parameters M and K control the width and depth the network, respectively. The second main contribution of our paper is Theorem 4.5, which shows that $\mathcal{S}_{\mathcal{F}}[M, K]$ is an effective feature extractor. Its proof uses Proposition 4.3, which shows that the largest coefficients of $\mathcal{S}_{\mathcal{F}}$ are concentrated in the frequency decreasing paths. It is unknown if $\mathcal{S}_{\mathcal{W}}$ satisfies this property.

In Section 5, we define the *fast Fourier scattering transform* algorithm, which computes $\mathcal{S}_{\mathcal{F}}[M, K]$. We use this algorithm to demonstrate what it intuitively does on a model image. Our experiment demonstrates that, while the first-order coefficients identifies the edges in the image, the second-order coefficients extract global and subtle oscillatory features.

In Section 6, we conclude this paper with a detailed comparison of our theoretical results with those from the literature. In particular, our proofs use covering arguments that have not been previously applied to the analysis of scattering transforms.

2. DEFINITIONS

Mathematically, a feature extractor is an operator, $S: X \rightarrow Y$, where X and Y are metric spaces. We primarily work with the data space $X = L^2(\mathbb{R}^d)$, the space of Lebesgue measurable functions that are square integrable, which provides an accurate model for audio and image data. We work with the feature space $Y = L^2(\mathbb{R}^d; \ell^2(\mathbb{Z}))$, the set of sequences $\{f_m: m \in \mathbb{Z}\}$ such that all $f_m: \mathbb{R}^d \rightarrow \mathbb{C}$ are Lebesgue measurable and

$$\|\{f_m\}\|_{L^2 \ell^2}^2 = \int_{\mathbb{R}^d} \sum_{m \in \mathbb{Z}} |f_m(x)|^2 dx < \infty.$$

In order to improve classification rates, an effective feature extractor S contracts distances between points belonging to the same class, and expands distances between points belonging to different classes; feature extractors that trivially contract or expand all data points are ineffective. For this reason, we want S to be bounded above and below. Otherwise, we can find sequences $\{f_n\}, \{g_n\} \subseteq X$ of unit norm vectors such that $\|Sf_n\|_Y \rightarrow 0$ and $\|Sg_n\|_Y \rightarrow \infty$.

Symmetries and invariants play an important role in feature extraction. For example, a small translation or perturbation of an image does not change its classification. More specifically, for $y \in \mathbb{R}^d$, let $|y|$ be its Euclidean norm and let $|y|_\infty$ be its sup norm. Let T_y be the translation operator

$$T_y f(x) = f(x - y).$$

Let $C^k(\mathbb{R}^d; \mathbb{R}^d)$ be the space of k -times continuous differentiable functions from \mathbb{R}^d to \mathbb{R}^d equipped with its usual norm $\|\cdot\|_{C^k}$. For $\tau \in C^1(\mathbb{R}^d; \mathbb{R}^d)$, let T_τ be the additive diffeomorphism

$$T_\tau f(x) = f(x - \tau(x)).$$

In order to demonstrate that S is an effective feature extractor, we would like to obtain finite upper bounds on $\|S(T_y f) - Sf\|_Y$ and $\|S(T_\tau f) - Sf\|_Y$ in terms of $|y|$, $\|\tau\|_{C^1}$, and $\|f\|_X$.

Mallat's windowed scattering transform [Mal12] satisfies variants of these properties, and we carefully discuss his results in Section 6. He constructed the windowed scattering transform by combining a specific wavelet frame with convolutional neural networks. Let J be an integer and let G be a finite group of rotations on \mathbb{R}^d together with reflection about the origin. Consider the wavelet frame,

$$\mathcal{W} = \{\varphi_{2^J}\} \cup \{\psi_{2^j, r}: j > -J, r \in G\},$$

where $\varphi_{2^J}(x) = 2^{-dJ} \varphi(2^{-J}x)$ is the wavelet corresponding to the coarsest scale 2^J , and $\psi_{2^j, r}(x) = 2^{dj} \psi(2^j r^{-1}x)$ is a detail wavelet of scale 2^{-j} and localization r . Here, we

have followed Mallat's notation where the dilations of φ and ψ are inversely related. The index set of \mathcal{W} is the countably infinite set

$$\Lambda = \{(2^j, r) : j > -J, r \in G\}.$$

The network structure is combined with the wavelet frame by creating a tree from the index set Λ and associating each element of the tree with a corresponding operator. Indeed, let $\Lambda^0 = \emptyset$, and for integers $k \geq 1$, let

$$\Lambda^k = \underbrace{\Lambda \times \Lambda \times \cdots \times \Lambda}_{k\text{-times}}.$$

Then, each $\lambda \in \Lambda^k$ is associated with the *scattering propagator* $U[\lambda]$, formally defined as

$$U[\lambda]f = \begin{cases} f & \text{if } \lambda \in \Lambda^0, \\ |f * \psi_\lambda| & \text{if } \lambda \in \Lambda, \\ U[\lambda_k]U[\lambda_{k-1}] \cdots U[\lambda_1]f & \text{if } \lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \Lambda^k. \end{cases}$$

Strictly speaking, it does not make sense to write $\lambda \in \Lambda^0 = \emptyset$, but we use this convention for convenience, see [Mal12]. The *windowed scattering transform* $\mathcal{S}_{\mathcal{W}}$ is formally defined as

$$\mathcal{S}_{\mathcal{W}}(f) = \{U[\lambda]f * \varphi_{2^j} : \lambda \in \Lambda^k, k = 0, 1, \dots\}.$$

Mallat's method for combining wavelets and neural networks is flexible, and we use his idea to combine neural networks with time-frequency representations called Gabor frames [BW93]. The Fourier transform of a Schwartz function f is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx,$$

and this definition has a unique extension to $f \in L^2(\mathbb{R}^d)$ by density. The essential support of a Lebesgue measurable function f , denoted $\text{supp}(f)$, is the complement of the largest open set where $f = 0$ almost everywhere.

Definition 2.1. A *uniform covering frame* is a sequence of functions,

$$\mathcal{F} = \{f_0\} \cup \{f_p : p \in \mathcal{P}\},$$

satisfying the following assumptions:

- (a) *Assumptions on f_0 and f_p .* Let \mathcal{P} be a countably infinite set. Let $f_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$ such that $\widehat{f_0}$ is supported in a neighborhood of the origin and $|\widehat{f_0}(0)| = 1$. For each $p \in \mathcal{P}$, let $f_p \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that $\text{supp}(\widehat{f_p})$ is compact and connected.
- (b) *Uniform covering property.* For any $R > 0$, there exists an integer $N > 0$ such that for each $p \in \mathcal{P}$, the set $\text{supp}(\widehat{f_p})$ can be covered by N cubes of side length $2R$.

(c) *Frame condition.* Assume that for all $\xi \in \mathbb{R}^d$,

$$|\widehat{f_0}(\xi)|^2 + \sum_{p \in \mathcal{P}} |\widehat{f_p}(\xi)|^2 = 1. \quad (2.1)$$

This implies \mathcal{F} is a semi-discrete Parseval frame for $L^2(\mathbb{R}^d)$: For all $f \in L^2(\mathbb{R}^d)$,

$$\|f * f_0\|_{L^2}^2 + \sum_{p \in \mathcal{P}} \|f * f_p\|_{L^2}^2 = \|f\|_{L^2}^2.$$

Remark 2.2. The uniform covering property is the key ingredient to our results, and we have several comments on this assumption.

- (a) The uniform covering property is, both, a size and a shape constraint on the sets $\{\text{supp}(\widehat{f_p}) : p \in \mathcal{P}\}$. It is a size constraint because it implies $\sup_{p \in \mathcal{P}} |\text{supp}(\widehat{f_p})| < \infty$, where $|S|$ denotes the Lebesgue measure of the set S . The uniform covering property is also a shape constraint because the number of cubes of a fixed side length required to cover the unit cube is much less than the number required to cover an elongated rectangular prism of unit volume.
- (b) Since a wavelet frame is partially generated by dilations of a single function, the support of each wavelet varies according to the dilation. Hence, no wavelet frame can satisfy the uniform covering property, and in turn, no wavelet frame can be a uniform covering frame. Examples of wavelet frames include standard wavelets [Dau92, Mal99], curvelets [CD04], shearlets [GKL06, GL07], composite wavelets [GLL⁺06], α -molecules [GKKS15], and Mallat's scattering wavelets [Mal12].
- (c) The assumption that $\text{supp}(\widehat{f_p})$ is connected is only used in Section 4, when we truncate $\mathcal{S}_{\mathcal{F}}$. The connectedness assumption is used to preclude certain pathological behavior such as $\text{supp}(\widehat{f_p})$ having two connected components, where one component is near the origin and the other is far from the origin.

We now explain why uniform covering frames are similar to Gabor frames. A Gabor frame covers the frequency space uniformly by translating a fixed set, while a uniform covering frame covers the frequency domain by sets of approximately equal size and shape. Hence, a uniform covering frame is similar to the time-frequency approach. In contrast to these approaches, a wavelet frame covers the frequency space non-uniformly by dilating a fixed set.

Given their similarities, it is not surprising that a variety of Gabor frames are uniform covering frames. It is possible to construct relevant and useful non-Gabor uniform covering frames; in the companion paper [CL16], we construct uniform covering frames that are partially generated by rotations to obtain a rotationally invariant operator, which we call the *rotational Fourier scattering transform*.

Proposition 2.3. *Let $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$ be such that $\text{supp}(\widehat{g})$ is compact and connected, $|\widehat{g}(0)| = 1$, and $\sum_{m \in \mathbb{Z}^d} |\widehat{g}(\xi - m)|^2 = 1$ for all $\xi \in \mathbb{R}^d$. Let $A: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an invertible linear transformation, $f_0(x) = |\det A| g(Ax)$, $\mathcal{P} = A^t \mathbb{Z}^d \setminus \{0\}$, and $f_p(x) = e^{2\pi i p \cdot x} f_0(x)$ for each $p \in \mathcal{P}$. Then, $\mathcal{F} = \{f_0\} \cup \{f_p : p \in \mathcal{P}\}$ is a Gabor frame, as well as a uniform covering frame.*

Proof. Since $\text{supp}(\widehat{f_p})$ is a translation of the connected and compact set $\text{supp}(\widehat{f_0})$, the uniform covering property automatically holds. Let $A^{-t} = (A^{-1})^t$. For all $\xi \in \mathbb{R}^d$,

$$|\widehat{f_0}(\xi)|^2 + \sum_{p \in \mathcal{P}} |\widehat{f_p}(\xi)|^2 = |\widehat{g}(A^{-t}\xi)|^2 + \sum_{p \in \mathcal{P}} |\widehat{g}(A^{-t}(\xi - p))|^2 = \sum_{m \in \mathbb{Z}^d} |\widehat{g}(A^{-t}\xi - m)|^2 = 1.$$

□

Figure 2.1 illustrates the differences between Mallat's wavelet frame \mathcal{W} [Mal12], and the Gabor frame that we just presented.

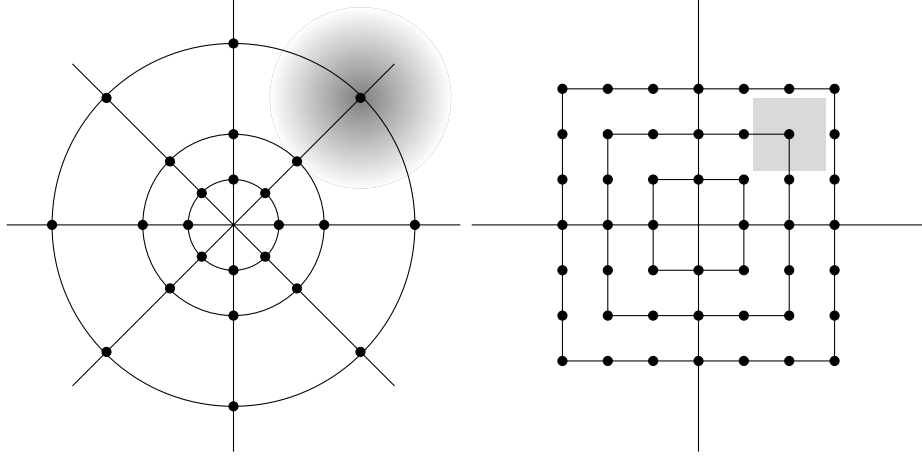


FIGURE 2.1. Left: Let G be the group of rotations by angle $2\pi/8$. The black dots are elements of Λ , for the first three dyadic scales. The shaded gray region is the effective support of $(\psi_{2^{-j+3}, 2\pi/8})^\wedge$. Right: Let A be the identity transformation. The black dots are elements of \mathcal{P} for the first three uniform Fourier scales. The shaded region is the support of $(f_{2,2})^\wedge$.

Having established the existence of a large class of uniform covering frames, we return our attention to incorporating the network structure. Slightly abusing notation, we associate $p \in \mathcal{P}^k$ with the scattering propagator $U[p]$, defined as

$$U[p]f = \begin{cases} f & \text{if } p \in \mathcal{P}^0, \\ |f * f_p| & \text{if } p \in \mathcal{P}, \\ U[p_k]U[p_{k-1}] \cdots U[p_1]f & \text{if } p = (p_1, p_2, \dots, p_k) \in \mathcal{P}^k. \end{cases}$$

Then, the *Fourier scattering transform*, $\mathcal{S}_{\mathcal{F}}$, is the vector-valued operator

$$\mathcal{S}_{\mathcal{F}}(f) = \{U[p]f * f_0 : p \in \mathcal{P}^k, k = 0, 1, \dots\}.$$

Since uniform covering frames decompose the frequency plane into approximately equal subsets, we believe “Fourier” is an appropriate description of this operator.

3. PROPERTIES OF THE FOURIER SCATTERING TRANSFORM

Since $\mathcal{S}_{\mathcal{W}}$ and $\mathcal{S}_{\mathcal{F}}$ have the same network structure, it is natural to ask whether they satisfy the same broad mathematical properties. This is not immediately clear because wavelets and Gabor functions are qualitatively and mathematically different, see [Dau90] for a comparison and discussion on applications. Despite their differences, we prove that $\mathcal{S}_{\mathcal{F}}$ satisfy all the same properties of $\mathcal{S}_{\mathcal{W}}$. However, our proof techniques are very different from those of Mallat's. For example, he used scaling and almost orthogonality arguments to exploit the dyadic structure of wavelets, while we use covering and tiling arguments to take advantage of the uniform covering property.

In order to show that $\mathcal{S}_{\mathcal{W}}$ and $\mathcal{S}_{\mathcal{F}}$ share the same properties, we need several preliminary results. For all $k \geq 0$ and $p \in \mathcal{P}^k$, it immediately follows from the frame property (2.1) that $U[p]: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is bounded with operator norm satisfying $\|U[p]\|_{L^2 \rightarrow L^2} \leq 1$. The following proposition contains some additional results that follow from the frame property as well. Mallat proved these for his wavelet frame \mathcal{W} in [Mal12], but the arguments only rely on the frame identity (2.1), so we omit their proofs.

Proposition 3.1. *Let $\mathcal{F} = \{f_0\} \cup \{f_p: p \in \mathcal{P}\}$ be a uniform covering frame. For $f, g \in L^2(\mathbb{R}^d)$ and integers $K \geq 0$, we have*

$$\sum_{p \in \mathcal{P}^{K+1}} \|U[p]f\|_{L^2}^2 + \sum_{k=0}^K \sum_{p \in \mathcal{P}^k} \|U[p]f * f_0\|_{L^2}^2 = \|f\|_{L^2}^2, \quad (3.1)$$

and

$$\sum_{k=0}^K \sum_{p \in \mathcal{P}^k} \|U[p]f * f_0 - U[p]g * f_0\|_{L^2}^2 \leq \|f - g\|_{L^2}^2. \quad (3.2)$$

The first identity of Proposition 3.1 implies that $\mathcal{S}_{\mathcal{F}}: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d; \ell^2(\mathbb{Z}))$ is bounded with operator norm satisfying $\|\mathcal{S}_{\mathcal{F}}\|_{L^2 \rightarrow L^2 \ell^2} \leq 1$. Indeed, we have

$$\|\mathcal{S}_{\mathcal{F}}(f)\|_{L^2 \ell^2}^2 = \lim_{K \rightarrow \infty} \sum_{k=0}^K \sum_{p \in \mathcal{P}^k} \|U[p]f * f_0\|_{L^2}^2 \leq \|f\|_{L^2}^2.$$

The following proposition is a basic result on positive definite functions. For any integer $k \geq 1$ and dimension d , Wendland [Wen95] constructed a compactly supported, radial, and positive-definite $C^{2k}(\mathbb{R}^d)$ function. These functions are essentially anti-derivatives of positive-definite polynomial splines. We provide a crude estimate since we do not need all of their smoothness. Let $Q_R(x) \subseteq \mathbb{R}^d$ denote the open cube of side length $2R > 0$ centered at $x \in \mathbb{R}^d$, namely, $Q_R(x) = \{y \in \mathbb{R}^d: |y - x|_{\infty} < R\}$.

Proposition 3.2. *There exists a non-negative function $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$, such that $\widehat{\phi}$ is continuous, decreasing along each Euclidean coordinate, and $\text{supp}(\widehat{\phi}) = \overline{Q_1(0)}$.*

Proof. For $j = 1, 2, \dots, d$, let $\phi_j: \mathbb{R} \rightarrow \mathbb{R}$ be defined by its one-dimensional Fourier transform,

$$\widehat{\phi}_j(\xi_j) = (1 - |\xi_j|)\mathbf{1}_{[0,1]}(|\xi_j|).$$

Here, $\mathbb{1}_S$ is the characteristic function of the set S and for a positive number x , $[x]$ stands for the integer n satisfying $n \leq x < n + 1$. Note that each ϕ_j is non-negative because $\widehat{\phi_j}$ is a univariate positive-definite function, see [Wen95]. Then, let $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$ be the function,

$$\phi(x) = \phi_1(x_1)\phi_2(x_2)\cdots\phi_d(x_d).$$

By construction, ϕ satisfies the desired properties. \square

The following proposition is the crucial exponential decay of energy estimate, and from here onwards, we let C_0 denote the constant that appears in the proposition.

Proposition 3.3. *Let $\mathcal{F} = \{f_0\} \cup \{f_p: p \in \mathcal{P}\}$ be a uniform covering frame. There exists a constant $C_0 \in (0, 1)$ depending only on \mathcal{F} , such that for all $f \in L^2(\mathbb{R}^d)$ and integers $K \geq 1$,*

$$C_0^{K-1} \|f * f_0\|_{L^2}^2 + \sum_{p \in \mathcal{P}^K} \|U[p]f\|_{L^2}^2 \leq C_0^{K-1} \|f\|_{L^2}^2.$$

Proof. By assumption, $\widehat{f_0}$ is continuous, supported in a neighborhood of the origin, and $|\widehat{f_0}(0)| = 1$. Then, by appropriately scaling the function discussed in Proposition 3.2, there exists a non-negative ϕ such that $\widehat{\phi}$ is continuous, decreasing along each Euclidean coordinate, $|\widehat{\phi}(0)| > 0$, and $|\widehat{\phi}(\xi)| \leq |\widehat{f_0}(\xi)|$ for all $\xi \in \mathbb{R}^d$. Then, there exist constants $R = R_\phi > 0$ and $C = C_\phi \in (0, 1)$, such that $|\widehat{\phi}(\xi)|^2 \geq C$ for all $\xi \in Q_R(0)$. By the uniform covering property, there exists an integer $N = N_R > 0$, such that for all $p \in \mathcal{P}$, there exist $\{\xi_{p,n} \in \mathbb{R}^d: n = 1, 2, \dots, N\}$ such that

$$\text{supp}(\widehat{f_p}) \subseteq \bigcup_{n=1}^N Q_R(\xi_{p,n}).$$

Let $1 \leq k \leq K$ and $q \in \mathcal{P}^{k-1}$. By Plancherel's formula and the above inclusion,

$$\|U[q]f * f_p\|_{L^2}^2 = \int_{\mathbb{R}^d} |\widehat{U[q]f}(\xi)|^2 |\widehat{f_p}(\xi)|^2 d\xi \leq \sum_{n=1}^N \int_{Q_R(\xi_{p,n})} |\widehat{U[q]f}(\xi)|^2 |\widehat{f_p}(\xi)|^2 d\xi.$$

Since $|\widehat{\phi}(\xi - \xi_{p,n})|^2 \geq C$ for all $\xi \in Q_R(\xi_{p,n})$, we have

$$\sum_{n=1}^N \int_{Q_R(\xi_{p,n})} |\widehat{U[q]f}(\xi)|^2 |\widehat{f_p}(\xi)|^2 d\xi \leq \frac{1}{C} \sum_{n=1}^N \int_{Q_R(\xi_{p,n})} |\widehat{U[q]f}(\xi)|^2 |\widehat{f_p}(\xi)|^2 |\widehat{\phi}(\xi - \xi_{p,n})|^2 d\xi.$$

By Plancherel's formula, we have

$$\sum_{n=1}^N \int_{Q_R(\xi_{p,n})} |\widehat{U[q]f}(\xi)|^2 |\widehat{f_p}(\xi)|^2 |\widehat{\phi}(\xi - \xi_{p,n})|^2 d\xi \leq \sum_{n=1}^N \|U[q]f * f_p * M_{\xi_{p,n}}\phi\|_{L^2}^2,$$

where M_y is the modulation operator, $M_y f(x) = e^{2\pi i y \cdot x} f(x)$. Using that $\phi \geq 0$ and triangle inequality, we have

$$\sum_{n=1}^N \|U[q]f * f_p * M_{\xi_{p,n}} \phi\|_{L^2}^2 \leq \sum_{n=1}^N \| |U[q]f * f_p| * \phi \|_{L^2}^2.$$

Observe that the terms inside the summation on the right hand side do not depend on the index n . Using Plancherel's formula and that $|\widehat{\phi}(\xi)| \leq |\widehat{f_0}(\xi)|$ for all $\xi \in \mathbb{R}^d$, we have

$$\| |U[q]f * f_p| * \phi \|_{L^2}^2 \leq \| |U[q]f * f_p| * f_0 \|_{L^2}^2.$$

Combining the previous inequalities and rearranging the result, we obtain

$$\| |U[q]f * f_p| * f_0 \|_{L^2}^2 \geq \frac{C}{N} \|U[q]f * f_p\|_{L^2}^2. \quad (3.3)$$

The strength of this inequality is that C and N are independent of $q \in \mathcal{P}^{k-1}$ and $p \in \mathcal{P}$. Then, summing this inequality over all $p \in \mathcal{P}$ and $q \in \mathcal{P}^{k-1}$, we see that

$$\sum_{p \in \mathcal{P}^k} \|U[p]f * f_0\|_{L^2}^2 \geq \frac{C}{N} \sum_{p \in \mathcal{P}^k} \|U[p]f\|_{L^2}^2.$$

Applying the frame identity (2.1) to the left hand side and setting $C_0 = 1 - C/N$, we have

$$\sum_{p \in \mathcal{P}^{k+1}} \|U[p]f\|_{L^2}^2 \leq C_0 \sum_{p \in \mathcal{P}^k} \|U[p]f\|_{L^2}^2.$$

Iterating the above inequality, we obtain

$$\sum_{p \in \mathcal{P}^K} \|U[p]f\|_{L^2}^2 \leq C_0^{K-1} \sum_{p \in \mathcal{P}} \|U[p]f\|_{L^2}^2 = C_0^{K-1} \|f\|_{L^2}^2 - C_0^{K-1} \|f * f_0\|_{L^2}^2.$$

□

Remark 3.4. The key step in the proof of Proposition 3.3 is to obtain the inequality (3.3), which is of the form,

$$\| |f * g| * f_0 \|_{L^2} \geq C \|f * g\|_{L^2},$$

for some constant $C > 0$ independent of $f, g \in L^2(\mathbb{R}^d)$. It is straightforward to obtain a lower bound where the constant depends on f and p . Indeed, suppose $f * g \neq 0$. Since $|f * g|$ is continuous, we have

$$(|f * g|)^\wedge(0) = \int_{\mathbb{R}^d} |(f * g)(x)| dx = \|f * g\|_{L^1} > 0.$$

By continuity of $|f * g|$, the above inequality, and the assumption that $|\widehat{f_0}(0)| = 1$, we can find a sufficiently small neighborhood $V = V_{f_0, f, g}$ of the origin and constants $C_{f_0}, C_{f, g} > 0$, such that $|\widehat{f_0}(\xi)| \geq C_{f_0}$ and $|(|f * g|)^\wedge(\xi)| \geq C_{f, g} \|f * g\|_{L^1}$ for all $\xi \in V$. Then,

$$\| |f * g| * f_0 \|_{L^2}^2 \geq \int_V |(|f * g|)^\wedge(\xi)|^2 |\widehat{f_0}(\xi)|^2 d\xi \geq C_{f_0} C_{f, g} \|f * g\|_{L^1} |V|,$$

where $|V|$ is the Lebesgue measure of V . However, both $C_{f,g}$ and $|V|$ depend on f and g , and $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ norms are not equivalent. Thus, this inequality is not very useful for our purposes. However, this naive reasoning suggests that a more sophisticated covering argument, such as the one given in the proof of Proposition 3.3, could work.

Remark 3.5. We have several comments about Proposition 3.3.

- (a) Since C_0 describes the rate of decay of $\sum_{p \in \mathcal{P}^k} \|U[p]f\|_{L^2}^2$, it is of interest to determine the optimal (smallest) value C_0 for which Proposition 3.3 holds. Minimizing C_0 is equivalent to maximizing the ratio C/N , where these constants were defined in the proof. Since N is related to the optimal covering by cubes of side length $2R$ and C is the minimum of $|\hat{\phi}|^2$ on $Q_R(0)$, both C and N decrease as R increases.
- (b) Let us examine why the argument proving inequality (3.3) fails for the wavelet case. Recall that $\psi_{2^j,r}$ has frequency scale 2^j whereas φ_{2^J} has frequency scale 2^{-J} . Using the same argument as in the proof of (3.3), we obtain: For each $j > -J$, there exists an integer $N_j > 0$ and $C_J \in (0, 1)$, such that for all $k \geq 1$, $p \in \Lambda^k$, $r \in G$, and $f \in L^2(\mathbb{R}^d)$,

$$\| |U[p]f * \psi_{2^j,r}| * \varphi_{2^J} \|_{L^2}^2 \geq \frac{C_J}{N_j} \|U[p]f * \psi_{2^j,r}\|_{L^2}^2.$$

The measure of $\text{supp}(\widehat{\psi_{2^j,r}})$ is proportional to 2^j , and N_j is the number of cubes required to cover this set with cubes of side length bounded by a constant multiple of 2^{-J} . Hence, $\lim_{j \rightarrow \infty} N_j = \infty$ and this inequality is not meaningful for large j .

- (c) Rather interestingly, numerical experiments in [Mal12, page 1345] have suggested that the exponential decay of energy described in the Proposition 3.3 holds for the wavelet case. Mallat conjectured that there exists $C \in (0, 1)$ such that

$$\sum_{\lambda \in \Lambda^K} \|U[\lambda]f\|_{L^2}^2 \leq C^{K-1} \|f\|_{L^2}^2,$$

for all $f \in L^2(\mathbb{R}^d)$, and $K \geq 1$. Determining whether this property holds for a given wavelet frame is of interest in the scattering community and Waldspurger [Wal15, Theorem 5.2, pages 145-146] proved a partial result of this nature for the wavelet case, but it required several additional assumptions.

We are ready to prove our first main theorem, which shows that $\mathcal{S}_{\mathcal{F}}$ satisfies several desirable properties as a feature extractor. For one of the results, we require mild additional regularity on f . We say that a function $f \in L^2$ is (ε, R) band-limited for some $\varepsilon \in [0, 1)$ and $R > 0$, if $\|\widehat{f}\|_{L^2(Q_R(0))} \geq (1 - \varepsilon)\|f\|_{L^2}$. Of course, if $\varepsilon = 0$, then f is band-limited. This assumption is realistic, since it has been observed that natural images are essentially band-limited [PM92].

Theorem 3.6. *Let $\mathcal{F} = \{f_0\} \cup \{f_p : p \in \mathcal{P}\}$ be a uniform covering frame, and let $\mathcal{S}_{\mathcal{F}}$ be the Fourier scattering transform.*

(a) $\mathcal{S}_{\mathcal{F}}$ conserves energy: For all $f \in L^2(\mathbb{R}^d)$,

$$\|\mathcal{S}_{\mathcal{F}}(f)\|_{L^2\ell^2} = \|f\|_{L^2}.$$

(b) $\mathcal{S}_{\mathcal{F}}$ is non-expansive on $L^2(\mathbb{R}^d)$: For all $f, g \in L^2(\mathbb{R}^d)$,

$$\|\mathcal{S}_{\mathcal{F}}(f) - \mathcal{S}_{\mathcal{F}}(g)\|_{L^2\ell^2} \leq \|f - g\|_{L^2}.$$

(c) $\mathcal{S}_{\mathcal{F}}$ contracts sufficiently small translations: There exists $C > 0$ depending only on \mathcal{F} , such that for all $f \in L^2(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$,

$$\|\mathcal{S}_{\mathcal{F}}(T_y f) - \mathcal{S}_{\mathcal{F}}(f)\|_{L^2\ell^2} \leq C|y|\|\nabla f_0\|_{L^1}\|f\|_{L^2}.$$

(d) $\mathcal{S}_{\mathcal{F}}$ contracts sufficiently small additive diffeomorphisms of almost band-limited functions: Let $\varepsilon \in [0, 1)$ and $R > 0$. There exists a universal constant $C > 0$, such that for all (ε, R) band-limited $f \in L^2(\mathbb{R}^d)$, and all $\tau \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ with $\|\nabla \tau\|_{L^\infty} \leq 1/(2d)$,

$$\|\mathcal{S}_{\mathcal{F}}(T_\tau f) - \mathcal{S}_{\mathcal{F}}(f)\|_{L^2\ell^2} \leq C(R\|\tau\|_{L^\infty} + \varepsilon)\|f\|_{L^2}.$$

Proof.

(a) Using identity (3.1) in Proposition 3.1 and Proposition 3.3, we obtain

$$\begin{aligned} \|\mathcal{S}_{\mathcal{F}}(f)\|_{L^2\ell^2}^2 &= \lim_{K \rightarrow \infty} \sum_{k=0}^K \sum_{p \in \mathcal{P}^k} \|U[p]f * f_0\|_{L^2}^2 \\ &= \|f\|_{L^2}^2 - \lim_{K \rightarrow \infty} \sum_{p \in \mathcal{P}^K} \|U[p]f\|_{L^2}^2 = \|f\|_{L^2}^2. \end{aligned}$$

(b) Using the identity (3.2) in Proposition 3.1, we obtain

$$\begin{aligned} \|\mathcal{S}_{\mathcal{F}}(f) - \mathcal{S}_{\mathcal{F}}(g)\|_{L^2\ell^2}^2 &= \lim_{K \rightarrow \infty} \sum_{k=0}^K \sum_{p \in \mathcal{P}^k} \|U[p]f * f_0 - U[p]g * f_0\|_{L^2}^2 \\ &\leq \|f - g\|_{L^2}^2. \end{aligned}$$

(c) By definition, we have

$$\|\mathcal{S}_{\mathcal{F}}(T_y f) - \mathcal{S}_{\mathcal{F}}(f)\|_{L^2\ell^2}^2 = \sum_{k=0}^{\infty} \sum_{p \in \mathcal{P}^k} \|U[p](T_y f) * f_0 - U[p]f * f_0\|_{L^2}^2.$$

Since translation commutes with convolution and the complex modulus, for all $k \geq 0$ and $p \in \mathcal{P}^k$, we have

$$U[p](T_y f) * f_0 = T_y(U[p]f) * f_0 = U[p]f * T_y f_0.$$

This fact, combined with Young's inequality, yields

$$\begin{aligned} \|U[p](T_y f) * f_0 - U[p]f * f_0\|_{L^2} &= \|U[p]f * (T_y f_0 - f_0)\|_{L^2} \\ &\leq \|T_y f_0 - f_0\|_{L^1} \|U[p]f\|_{L^2}. \end{aligned}$$

Then, we have

$$\|\mathcal{S}_{\mathcal{F}}(T_y f) - \mathcal{S}_{\mathcal{F}}(f)\|_{L^2 \ell^2} \leq \|T_y f_0 - f_0\|_{L^1} \left(\sum_{k=0}^{\infty} \sum_{p \in \mathcal{P}^k} \|U[p]f\|_{L^2}^2 \right)^{1/2}. \quad (3.4)$$

We first bound the summation in (3.4). Using Proposition 3.3, we have

$$\sum_{k=0}^{\infty} \sum_{p \in \mathcal{P}^k} \|U[p]f\|_{L^2}^2 \leq \|f\|_{L^2}^2 + \sum_{k=1}^{\infty} C_0^{k-1} \|f\|_{L^2}^2 = \left(1 + \frac{1}{1 - C_0}\right) \|f\|_{L^2}^2.$$

To bound the L^1 term in (3.4), we use the fundamental theorem of calculus, which is justified by the assumption that $f_0 \in C^1(\mathbb{R}^d)$. Then, we obtain

$$\int_{\mathbb{R}^d} |f_0(x - y) - f_0(x)| \, dx = \int_{\mathbb{R}^d} \left| \int_0^1 \nabla f_0(x - ty) \cdot y \, dt \right| \, dx \leq |y| \|\nabla f_0\|_{L^1}.$$

- (d) Let ϕ be a Schwartz function such that $\widehat{\phi}$ is real-valued, supported in $\overline{Q_2(0)}$, and $\widehat{\phi}(\xi) = 1$ for all $\xi \in Q_1(0)$. Let $\phi_R(x) = R^d \phi(Rx)$, and let $f_R = f * \phi_R$. By the non-expansiveness property and triangle inequality

$$\|\mathcal{S}_{\mathcal{F}}(T_{\tau} f) - \mathcal{S}_{\mathcal{F}}(f)\|_{L^2 \ell^2} \leq \|f - f_R\|_{L^2} + \|T_{\tau}(f_R) - f_R\|_{L^2} + \|T_{\tau}(f_R) - T_{\tau} f\|_{L^2}. \quad (3.5)$$

The bound for the first term in (3.5) follows by assumption

$$\|f - f_R\|_{L^2} \leq \varepsilon \|f\|_{L^2}.$$

To bound the second term in (3.5), we make the change of variable $u = x - \tau(x)$ and note that

$$\left| \frac{\partial u}{\partial x} \right| = |\det(I - \nabla \tau(x))| \geq (1 - d \|\nabla \tau\|_{L^\infty}) \geq \frac{1}{2},$$

see [BOS15]. Then, we have

$$\begin{aligned} \|T_{\tau}(f_R) - T_{\tau} f\|_{L^2}^2 &= \int_{\mathbb{R}^d} |f_R(x - \tau(x)) - f(x - \tau(x))|^2 \, dx \\ &\leq 2 \|f - f_R\|_{L^2}^2 \\ &\leq 2\varepsilon^2 \|f\|_{L^2}^2. \end{aligned}$$

It remains to bound the third term of (3.5), and we use the argument proved in [WB15, Proposition 5]. We have

$$\begin{aligned} (T_{\tau} f_R)(x) - f_R(x) &= (f * \phi_R)(x - \tau(x)) - (f * \phi_R)(x) \\ &= \int_{\mathbb{R}^d} (\phi_R(x - \tau(x) - y) - \phi_R(x - y)) f(y) \, dy. \end{aligned}$$

The above can be interpreted as an integral kernel operator acting on $f \in L^2(\mathbb{R}^d)$ with kernel

$$k(x, y) = \phi_R(x - \tau(x) - y) - \phi_R(x - y).$$

The proof is completed by verifying that this kernel satisfies the assumptions of Schur's lemma with the appropriate bounds. By the fundamental theorem of calculus, we have

$$\begin{aligned} |k(x, y)| &= \left| \int_0^1 \nabla \phi_R(x - t\tau(x) - y) \cdot \tau(x) dt \right| \\ &\leq \|\tau\|_{L^\infty} \int_0^1 |\nabla \phi_R(x - t\tau(x) - y)| dt. \end{aligned}$$

(i) For each $x \in \mathbb{R}^d$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} |k(x, y)| dy &\leq \|\tau\|_{L^\infty} \int_0^1 \int_{\mathbb{R}^d} |\nabla \phi_R(x - t\tau(x) - y)| dy dt \\ &= R \|\nabla \phi\|_{L^1} \|\tau\|_{L^\infty}. \end{aligned}$$

(ii) For each $y \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} |k(x, y)| dx \leq \|\tau\|_{L^\infty} \int_0^1 \int_{\mathbb{R}^d} |\nabla \phi_R(x - t\tau(x) - y)| dx dt.$$

For fixed $y \in \mathbb{R}^d$ and $t \in [0, 1]$, we make the change of variables $v = x - t\tau(x) - y$ and note that

$$\left| \frac{\partial v}{\partial x} \right| = |\det(I - t\nabla \tau(x))| \geq (1 - t\|\nabla \tau\|_{L^\infty}) \geq \frac{1}{2}.$$

Thus, for all $y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |k(x, y)| dx \leq 2\|\tau\|_{L^\infty} \int_0^1 \int_{\mathbb{R}^d} |\nabla \phi_R(v)| dv dt = 2R \|\nabla \phi\|_{L^1} \|\tau\|_{L^\infty}.$$

By Schur's lemma, we conclude that

$$\|T_\tau f - f\|_{L^2} \leq 2R \|\nabla \phi\|_{L^1} \|\tau\|_{L^\infty} \|f\|_{L^2}.$$

Combining the above results, we obtain the inequality

$$\|\mathcal{S}_\mathcal{F}(T_\tau f) - \mathcal{S}_\mathcal{F}(f)\|_{L^2 \ell^2} \leq (2R \|\nabla \phi\|_{L^1} \|\tau\|_{L^\infty} + \varepsilon + \sqrt{2}\varepsilon) \|f\|_{L^2}.$$

Set $C = \max(2\|\nabla \phi\|_{L^1}, 1 + \sqrt{2})$, which completes the proof. \square

4. TRUNCATED FOURIER SCATTERING TRANSFORM

Both $\mathcal{S}_\mathcal{F}$ and $\mathcal{S}_\mathcal{W}$ correspond to neural networks of infinite width and depth, but when used in practice, the networks must be truncated. To truncate $\mathcal{S}_\mathcal{F}$, we keep terms up to a certain depth K and terms belonging to appropriate finite subsets of \mathcal{P}^k , for $k = 1, 2, \dots, K$.

However, it is not clear whether their truncated versions are non-trivial and satisfy the same properties that we previously mentioned. For example, let us focus our attention on the first layer. Say we compute a finite number of first-order coefficients,

$$\{|f * f_p| * f_0 : p \in S\},$$

for some finite subset $S \subseteq \mathcal{P}$. However, we can always find a non-trivial $f \in L^2(\mathbb{R}^d)$ such that $f * f_p = 0$ for all $p \in S$. This is already problematic, since it shows that this truncated operator has non-trivial kernel and consequently, is not bounded from below. This shows that, in order to truncate just the first layer of coefficients, we need an additional assumption on $f \in L^2(\mathbb{R}^d)$. The most natural assumption is that f is (ε, R) band-limited, and then S can be chosen appropriately depending on R .

Now, we focus our attention on the higher-order coefficients. The naive idea is to use the truncation $S^k \subseteq \mathcal{P}^k$ and compute the finite set of coefficients

$$\{U[p]f * f_0 : p \in S^k, k = 0, 1, \dots, K\}.$$

This truncation tosses away the high frequency terms and might seem reasonable since Proposition 3.3 showed that the complex modulus pushes higher frequencies to lower frequencies. Indeed, for $f \in L^2(\mathbb{R}^d)$ and $p \in \mathcal{P}$, the proposition showed that $(|f * f_p|)^\wedge$ is non-zero in a neighborhood of the origin even though $(f * f_p)^\wedge$ is compactly supported away from the origin. However, the complex modulus can also push lower frequencies to higher frequencies. To see why, we note that the function $|f * f_p|$ is continuous, but in general, it is not $C^1(\mathbb{R}^d)$; even if we make the very mild assumption that $\nabla f_p \in L^1(\mathbb{R}^d)$, we can only conclude that $|f * f_p|$ has one distributional derivative belonging to $L^2(\mathbb{R}^d)$. Thus, the decay of $(|f * f_p|)^\wedge$ is quite slow, even though $(f * f_p)^\wedge$ is compactly supported! This observation shows that we must be careful when truncating $\mathcal{S}_{\mathcal{F}}$. However, we shall see that this naive truncation in fact works, but only requires an alternative argument.

We first prove the following proposition, which relates the index $p \in \mathcal{P}$ with the location of $\text{supp}(\widehat{f}_p)$. Observe that (2.1) is a partition of unity statement, but it does not provide any information on how the partitioning is structured. Not surprisingly, the partition of unity has to be done in a “uniform” way due to the uniform covering property.

Proposition 4.1. *Let $\mathcal{F} = \{f_0\} \cup \{f_p : p \in \mathcal{P}\}$ be a uniform covering frame. There exist a constant $C_1 > 0$ and subsets $\{\mathcal{P}[m] \subseteq \mathcal{P} : m \geq 1\}$ such that for all integers $m \geq 1$,*

$$|\widehat{f}_0(\xi)|^2 + \sum_{p \in \mathcal{P}[m]} |\widehat{f}_p(\xi)|^2 = \begin{cases} 1 & \text{if } \xi \in \overline{Q_{C_1 m}(0)}, \\ 0 & \text{if } \xi \notin Q_{C_1(m+1)}(0). \end{cases} \quad (4.1)$$

Proof. For any set $S \subseteq \mathbb{R}^d$, let $\text{diam}(S) = \sup_{x, y \in S} |x - y|$ be the diameter of S . Define

$$C_1 = \max \left(\text{diam}(\text{supp}(\widehat{f}_0)), \sup_{p \in \mathcal{P}} \text{diam}(\text{supp}(\widehat{f}_p)) \right).$$

Note that C_1 is finite because of the uniform covering property. Indeed, the diameter of $\text{supp}(\widehat{f}_0)$ is finite since \widehat{f}_0 is supported in a compact set containing the origin. Fix $R > 0$, and by assumption, the closed and connected set $\text{supp}(\widehat{f}_p)$ can be covered by N cubes of side length $2R$. Then, the diameter of $\text{supp}(\widehat{f}_p)$ is bounded by $2NR$.

For integers $m \geq 1$, we define

$$\mathcal{P}[m] = \{p \in \mathcal{P} : \text{supp}(\widehat{f}_p) \subseteq \overline{Q_{C_1(m+1)}(0)}\}.$$

By definition, of $\mathcal{P}[m]$, we have

$$|\widehat{f}_0(\xi)|^2 + \sum_{p \in \mathcal{P}[m]} |\widehat{f}_p(\xi)|^2 = 0 \quad \text{if } \xi \notin Q_{C_1(m+1)}(0).$$

To complete the proof, we prove (4.1) by contradiction. Suppose there exists $\xi_0 \in Q_{C_1 m}(0)$ such that

$$|\widehat{f}_0(\xi_0)|^2 + \sum_{p \in \mathcal{P}[m]} |\widehat{f}_p(\xi_0)|^2 < 1.$$

By the frame condition (2.1), there exists $q \in \mathcal{P}$ such that $|\widehat{f}_q(\xi_0)| > 0$. Then, $\xi_0 \in \text{supp}(\widehat{f}_q)$ and by definition of $C_1 > 0$, we have

$$\text{supp}(\widehat{f}_q) \subseteq \overline{Q_{C_1}(\xi_0)} \subseteq \overline{Q_{C_1(m+1)}(0)}. \quad (4.2)$$

For an illustration of this inclusion, see Figure 4.1. This shows that $q \in \mathcal{P}[m]$, which contradicts the definition of $\mathcal{P}[m]$.

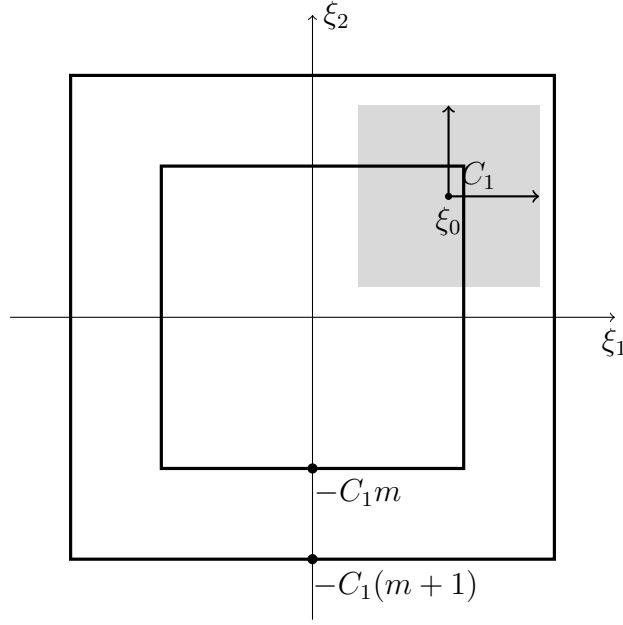


FIGURE 4.1. An illustration of the inclusions (4.2).

□

From here onwards, let $C_1 > 0$ be the smallest constant such that Proposition 4.1 holds, and let $\{\mathcal{P}[m]: m \geq 1\}$ be the family of sets defined in the proposition.

Remark 4.2. Suppose \mathcal{F} is a Gabor frame satisfying Proposition 2.3 for $A = aI$, where I is the identity transformation on \mathbb{R}^d and $a > 0$. By definition, we have $\mathcal{P} = a\mathbb{Z}^d \setminus \{0\}$. We can determine the family of sets $\{\mathcal{P}[m]: m \geq 1\}$ satisfying Proposition 4.1. Let $C_1 = a$ and

$$\mathcal{P}[m] = \{p \in \mathcal{P}: |p|_\infty \leq m\}.$$

For all integers $m \geq 1$, we have

$$|\widehat{f}_0(\xi)|^2 + \sum_{p \in \mathcal{P}[m]} |\widehat{f}_p(\xi)|^2 = \begin{cases} 1 & \text{if } \xi \in \overline{Q_{am}(0)}, \\ 0 & \text{if } \xi \notin \overline{Q_{a(m+1)}(0)}. \end{cases}$$

To define the truncated Fourier scattering transform, we create a tree from the finite index set $\mathcal{P}[M]$. For integers $M, K \geq 1$, we define the discrete set

$$\mathcal{P}[M]^K = \underbrace{\mathcal{P}[M] \times \mathcal{P}[M] \times \cdots \times \mathcal{P}[M]}_{K\text{-times}} \subseteq \mathcal{P}^K. \quad (4.3)$$

Again, we use the convention that $\mathcal{P}[M]^0 = \emptyset$. Then, the *truncated Fourier scattering transform*, $\mathcal{S}_{\mathcal{F}}[M, K]: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d; \ell^2(\mathbb{Z}))$, is defined as

$$\mathcal{S}_{\mathcal{F}}[M, K](f) = \{U[p]f * f_0: p \in \mathcal{P}[M]^k, k = 0, 1, \dots, K\}. \quad (4.4)$$

To demonstrate that this truncation is acceptable, we are concerned with bounding terms of the form, $\| |f * f_p| * f_q \|_{L^2}$, where $p \in \mathcal{P}[M]$ and $q \in \mathcal{P}[M]^c$. These are the terms that are thrown away due to truncation, and since f_p has lower frequencies than f_q , we expect them to be small. The following proposition shows that this intuition holds. That is, most of the energy is concentrated along the frequency decaying paths.

Proposition 4.3. *Let $\mathcal{F} = \{f_0\} \cup \{f_p: p \in \mathcal{P}\}$ be a uniform covering frame. For any integer $M \geq 1$, there exists $C_M \in (0, 1)$ such that $C_M \rightarrow 1$ as $M \rightarrow \infty$ and for all integers $k \geq 1$, $p \in \mathcal{P}^k$, and $f \in L^2(\mathbb{R}^d)$,*

$$\|U[p]f * f_0\|_{L^2}^2 + \sum_{q \in \mathcal{P}[M]} \|U[p]f * f_q\|_{L^2}^2 \geq C_M \|U[p]f\|_{L^2}^2.$$

Proof. By Proposition 3.2, there exists a non-negative function ϕ , such that $\widehat{\phi}$ is continuous, decreasing along each Euclidean coordinate, $\text{supp}(\widehat{\phi}) = \overline{Q_1(0)}$, and $|\widehat{\phi}(0)| = 1$. Define ϕ_M by its Fourier transform, $\widehat{\phi}_M(\xi) = \widehat{\phi}(C_1^{-1}M^{-1}\xi)$, for $\xi \in \mathbb{R}^d$. By definition of $\mathcal{P}[M]$, for all $\xi \in \mathbb{R}^d$,

$$|\widehat{\phi}_M(\xi)|^2 \leq |\widehat{f}_0(\xi)|^2 + \sum_{q \in \mathcal{P}[M]} |\widehat{f}_q(\xi)|^2.$$

Plancherel's formula and this inequality imply

$$\begin{aligned} \|U[p]f * f_0\|_{L^2}^2 + \sum_{q \in \mathcal{P}[M]} \|U[p]f * f_q\|_{L^2}^2 &\geq \|U[p]f * \phi_M\|_{L^2}^2 \\ &= \| |U[p']f * f_s| * \phi_M \|_{L^2}^2, \end{aligned} \quad (4.5)$$

where $p = (p', s)$. By definition of C_1 , there exists $\xi_s \in \mathbb{R}^d$ such that $\text{supp}(\widehat{f}_s) \subseteq \overline{Q_{C_1/2}(\xi_s)}$. Applying triangle inequality to the right hand side of (4.5) and using that $\phi \geq 0$, we have

$$\begin{aligned} \| |U[p']f * f_s| * \phi_M \|_{L^2}^2 &\geq \|U[p']f * f_s * M_{\xi_s} \phi_M\|_{L^2}^2 \\ &= \int_{\mathbb{R}^d} |\widehat{U[p']f}(\xi)|^2 |\widehat{f}_s(\xi)|^2 |\widehat{\phi}_M(\xi - \xi_s)|^2 d\xi. \end{aligned} \quad (4.6)$$

Using that $\widehat{\phi}$ is decreasing along each Euclidean coordinate and the inclusion $\text{supp}(\widehat{f_s}) \subseteq \overline{Q_{C_1/2}(\xi_s)}$, we have

$$\begin{aligned} C_M &= \inf_{\xi \in \text{supp}(\widehat{f_s})} |\widehat{\phi_M}(\xi - \xi_s)|^2 \\ &\geq \inf_{\xi \in \overline{Q_{C_1/2}(0)}} |\widehat{\phi_M}(\xi)|^2 \\ &= |\widehat{\phi}(2^{-1}M^{-1}, 2^{-1}M^{-1}, \dots, 2^{-1}M^{-1})|^2 > 0. \end{aligned}$$

Inserting this into (4.6) and applying Plancherel's formula yields

$$\int_{\mathbb{R}^d} |\widehat{U[p']f}(\xi)|^2 |\widehat{f_s}(\xi)|^2 |\widehat{\phi_M}(\xi - \xi_s)|^2 d\xi \geq C_M \|U[p']f * f_s\|_{L^2}^2 = C_M \|U[p]f\|_{L^2}^2.$$

We have the trivial inequality $C_M \leq 1$, and observe that

$$\liminf_{M \rightarrow \infty} C_M \geq \liminf_{M \rightarrow \infty} |\widehat{\phi}(2^{-1}M^{-1}, 2^{-1}M^{-1}, \dots, 2^{-1}M^{-1})|^2 = |\widehat{\phi}(0)|^2 = 1.$$

□

Remark 4.4. This argument fails for wavelet frames. Indeed, we made use of the uniform tiling property in Proposition 4.1 and that $\widehat{f_p}$ is supported in a cube of side length C_1 , where C_1 is independent of $p \in \mathcal{P}$.

We are ready to prove our second main theorem, which shows that $\mathcal{S}_{\mathcal{F}}[M, K]$ is an effective feature extractor.

Theorem 4.5. *Let $\mathcal{F} = \{f_0\} \cup \{f_p : p \in \mathcal{P}\}$ be a uniform covering frame. Fix integers $M, K \geq 1$, and let $\mathcal{S}_{\mathcal{F}}[M, K]$ be the truncated Fourier scattering transform.*

(a) $\mathcal{S}_{\mathcal{F}}[M, K]$ satisfies the upper bound: For all $f \in L^2(\mathbb{R}^d)$,

$$\|\mathcal{S}_{\mathcal{F}}[M, K](f)\|_{L^2 \ell^2} \leq \|f\|_{L^2}.$$

(b) $\mathcal{S}_{\mathcal{F}}[M, K]$ satisfies the lower bound for almost band-limited functions: Let $\varepsilon \in [0, 1)$ and $R > 0$. There exist integers $K \geq 1$ and $M \geq C_1^{-1}R$ sufficiently large depending on ε , such that for all (ε, R) band-limit functions $f \in L^2(\mathbb{R}^d)$,

$$\|\mathcal{S}_{\mathcal{F}}[M, K](f)\|_{L^2 \ell^2}^2 \geq (C_M^K(1 - \varepsilon^2) - C_0^{K-1})\|f\|_{L^2}^2.$$

(c) $\mathcal{S}_{\mathcal{F}}[M, K]$ is non-expansive on $L^2(\mathbb{R}^d)$: For all $f, g \in L^2(\mathbb{R}^d)$,

$$\|\mathcal{S}_{\mathcal{F}}[M, K](f) - \mathcal{S}_{\mathcal{F}}[M, K](g)\|_{L^2 \ell^2} \leq \|f - g\|_{L^2}.$$

(d) $\mathcal{S}_{\mathcal{F}}[M, K]$ contracts sufficiently small translations of $L^2(\mathbb{R}^d)$: There exists a constant $C > 0$ depending only on \mathcal{F} such that for all $f \in L^2(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, we have

$$\|\mathcal{S}_{\mathcal{F}}[M, K](T_y f) - \mathcal{S}_{\mathcal{F}}[M, K](f)\|_{L^2 \ell^2} \leq C|y| \|\nabla f_0\|_{L^1} \|f\|_{L^2}.$$

- (e) $\mathcal{S}_{\mathcal{F}}[M, K]$ contracts sufficiently small additive diffeomorphisms of almost band-limited functions: Let $\varepsilon \in [0, 1)$ and $R > 0$. There exists a universal constant $C > 0$, such that for all (ε, R) band-limited $f \in L^2(\mathbb{R}^d)$, and all $\tau \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ with $\|\nabla \tau\|_{L^\infty} \leq 1/(2d)$,

$$\|\mathcal{S}_{\mathcal{F}}[M, K](T_\tau f) - \mathcal{S}_{\mathcal{F}}[M, K](f)\|_{L^2 \ell^2} \leq C(R\|\tau\|_{L^\infty} + \varepsilon)\|f\|_{L^2}.$$

Proof.

- (a) We apply Theorem 3.6 to obtain,

$$\|\mathcal{S}_{\mathcal{F}}[M, K](f)\|_{L^2 \ell^2} \leq \|\mathcal{S}_{\mathcal{F}}(f)\|_{L^2 \ell^2} = \|f\|_{L^2}.$$

- (b) By the tiling property (4.1), the assumption that f is almost band-limited, and that $C_1 M \geq R$, we have

$$\|f\|_{L^2}^2 = \|f * f_0\|_{L^2}^2 + \sum_{p \in \mathcal{P}[M]} \|f * f_p\|_{L^2}^2 + \varepsilon^2 \|f\|_{L^2}^2.$$

Applying Proposition 4.3 to the summation over $\mathcal{P}[M]$, we obtain,

$$(1 - \varepsilon^2)\|f\|_{L^2}^2 \leq \|f * f_0\|_{L^2}^2 + C_M^{-1} \sum_{p \in \mathcal{P}[M]} \|U[p]f * f_0\|_{L^2}^2 + C_M^{-1} \sum_{p \in \mathcal{P}[M]^2} \|U[p]f\|_{L^2}^2.$$

Continuing to apply Proposition 4.3, we see that

$$(1 - \varepsilon^2)\|f\|_{L^2}^2 \leq \sum_{k=0}^K C_M^{-k} \sum_{p \in \mathcal{P}[M]^k} \|U[p]f * f_0\|_{L^2}^2 + C_M^{-K} \sum_{p \in \mathcal{P}[M]^K} \|U[p]f\|_{L^2}^2.$$

Using that $C_M \in (0, 1)$ and Proposition 3.3, we have

$$(1 - \varepsilon^2)\|f\|_{L^2}^2 \leq C_M^{-K} \sum_{k=0}^K \sum_{p \in \mathcal{P}[M]^k} \|U[p]f * f_0\|_{L^2}^2 + C_M^{-K} C_0^{K-1} \|f\|_{L^2}^2.$$

Rearranging, we obtain

$$\|\mathcal{S}_{\mathcal{F}}[M, K](f)\|_{L^2 \ell^2}^2 = \sum_{k=0}^K \sum_{p \in \mathcal{P}[M]^k} \|U[p]f * f_0\|_{L^2}^2 \geq (C_M^K (1 - \varepsilon^2) - C_0^{K-1}) \|f\|_{L^2}^2.$$

Since $C_M \rightarrow 1$ as $M \rightarrow \infty$ and $C_0 \in (0, 1)$ independent of M , for fixed ε , we can pick K and M sufficiently large so that $C_M^K (1 - \varepsilon^2) - C_0^{K-1} > 0$. Note that this term represents the error due to approximating f by a band-limited function, the horizontal truncation, and the depth truncation.

- (c)-(e) For any $f, g \in L^2(\mathbb{R}^d)$, we have

$$\|\mathcal{S}_{\mathcal{F}}[M, K](f) - \mathcal{S}_{\mathcal{F}}[M, K](g)\|_{L^2 \ell^2}^2 \leq \|\mathcal{S}_{\mathcal{F}}(f) - \mathcal{S}_{\mathcal{F}}(g)\|_{L^2 \ell^2}^2.$$

Applying Theorem 3.6 completes the proof. □

5. A MOTIVATIONAL EXPERIMENT

We write down a simple algorithm that computes $\mathcal{S}_{\mathcal{F}}[M, K](f)$ for an input f , and we call this algorithm the *fast Fourier scattering transform*.

Algorithm 5.1. Fast Fourier scattering transform.

Input: Function f , network depth $K \geq 1$, network width $M \geq 1$
Construct: Frame elements $\{f_0\} \cup \{f_p: p \in \mathcal{P}[M]\}$
for $k = 1, 2, \dots, K$
 for each $p = (p', p_k) \in \mathcal{P}[M]^k$
 Compute $U[p]f = U[(p', p_k)]f = |U[p']f * f_{p_k}|$ and $U[p]f * f_0$
 end
end

Remark 5.2. The reason we call this algorithm fast is because Theorem 4.5 quantifies the amount of information lost due to truncation, so we do not have to calculate an enormous number of scattering coefficients. In fact, it is possible to make the algorithm even faster:

- (a) In many applications, the input function f is real-valued. Suppose that for each $p \in \mathcal{P}[M]$, $\widehat{f_p}$ is real-valued and there exists a unique $q \in \mathcal{P}[M]$ such that $\widehat{f_p}(\xi) = \widehat{f_q}(-\xi)$ for all $\xi \in \mathbb{R}^d$. Then, we have $|f * f_p| = |f * f_q|$. For any $k \geq 1$ and $p \in \mathcal{P}^k$, $U[p]f$ is also real-valued, so the same reasoning shows that for all $k \geq 1$ and $p \in \mathcal{P}[M]^k$, there exists a unique $q \in \mathcal{P}[M]^k$ such that $U[p]f = U[q]f$. Thus, we only need to compute half of the coefficients in the fast Fourier scattering transform. These assumptions hold, for example, if \mathcal{F} is a Gabor frame defined in Proposition 2.3 and the window $\widehat{f_0} = \widehat{g}$ is real-valued and symmetric about the origin.
- (b) The proof of Proposition 4.3 shows that coefficients of the form $|f * f_p| * f_q|$ have small $L^2(\mathbb{R}^d)$ norm whenever $p \in \mathcal{P}[M]$, $q \in \mathcal{P}[N]$, and $M \ll N$. Since our algorithm still computes such coefficients, its runtime can be greatly reduced by not computing these coefficients.

We provide a numerical example that demonstrates what the Fourier scattering coefficients look like. The Matlab code generating the results can be found on [Li16]. For this experiment, we use a Gabor frame \mathcal{F} satisfying Proposition 2.3, where $a = 16$, $A = aI$, $\text{supp}(\widehat{g}) = [-a, a]^2$, and \widehat{g} is real-valued and even. Note that the family of sets $\{\mathcal{P}[m]: m \geq 1\}$ were defined in Remark 4.2. We use the popular Lena image as our model example. We interpret the square $N \times N$ image as samples of a function f , namely

$$\{f(m_1, m_2): m_1 = 1, 2, \dots, N, m_2 = 1, 2, \dots, N\}.$$

We set $M = K = 2$ and compute $\mathcal{S}_{\mathcal{F}}[M, K](f)$ using the fast Fourier scattering transform. In total, we generate one zero-order coefficient, 12 first-order coefficients, and 144 second-order coefficients. While it is interesting to look at each coefficient, there are too many to display. Figure 5.1 displays all of the coefficients with $L^2(\mathbb{R}^d)$

norm greater than $0.01 * \|f\|_{L^2}$. While these have the largest norm, we do not claim that these are the only important coefficients.

As seen in the figure, the first-order coefficients extract distinct features of the image, and in particular, extract the most prominent edges in the Lena image. For example, the fifth image contains edges that are oriented in the north-east direction and the sixth image contains the round edges such as her shoulder and the top of her hat. The second-order coefficients capture more subtle oscillatory behavior that are not present in the first-order coefficients. Moreover, the features captured in the first-order coefficients are well-localized, whereas the features captured in the second-order coefficients are not.

In general, it is difficult to substantiate what functions of the form $| |f * f_p| * f_q|$ intuitively mean. This is partly because the Fourier transform is the standard tool for analyzing convolutions, but is not well suited for handling non-linear operators such as the complex modulus. For the wavelet case, Mallat has heuristically argued that coefficients of the form $| |f * \psi_{2^j, r}| * \psi_{2^k, s}|$ describe interactions between scales 2^{-j} and 2^{-k} [Mal16]. By the same reasoning, coefficients of the form $| |f * f_p| * f_q|$ describe the interactions between oscillations arising from the uniform Fourier scales $|p|_\infty$ and $|q|_\infty$.

6. DISCUSSION

In addition to the papers and PhD theses from Mallat's group [Mal12, BM13, Wal15], we note Wiatowski and Bölcskei [WB15] is another mathematical work on Mallat scattering transforms. In this section, we compare our results with theirs, in the following ways.

- (a) *Generality and flexibility.* Wiatowski and Bölcskei studied a scattering framework that is more general than ours and Mallat's. Instead of using the same semi-discrete wavelet frame for each layer of the network, they used (not necessarily tight) semi-discrete frames, and allowed each layer of the transformation to use a different frame. Their theory allowed for a variety of non-linearities at each layer, including the complex modulus, and allowed sub-sampling to be incorporated into each layer. They called their resulting scattering transform the *generalized feature extractor* Φ . However, as we shall see, their results are weaker in several ways.

Mallat's results on $\mathcal{S}_{\mathcal{W}}$ require mild assumptions on \mathcal{W} , but several of his important results require a restrictive and technical *admissibility condition* on ψ , see [Mal12, pages 1342-1343]. Note that this is not related to the usual admissibility condition related to the invertibility of the continuous wavelet transform. We cannot offer an intuitive explanation for what Mallat's complicated admissibility condition means. Numerical calculations have supported the assertion that an analytic cubic spline Battle-Lemarié wavelet is admissible for $d = 1$, [Mal12, page 1345]. To our best knowledge, it is currently unknown if other Littlewood-Paley wavelets, such as curvelets [CD04] or shearlets [GKL06, GL07], are admissible. Hence, the majority of Mallat's theory is only known to hold for this specific wavelet.

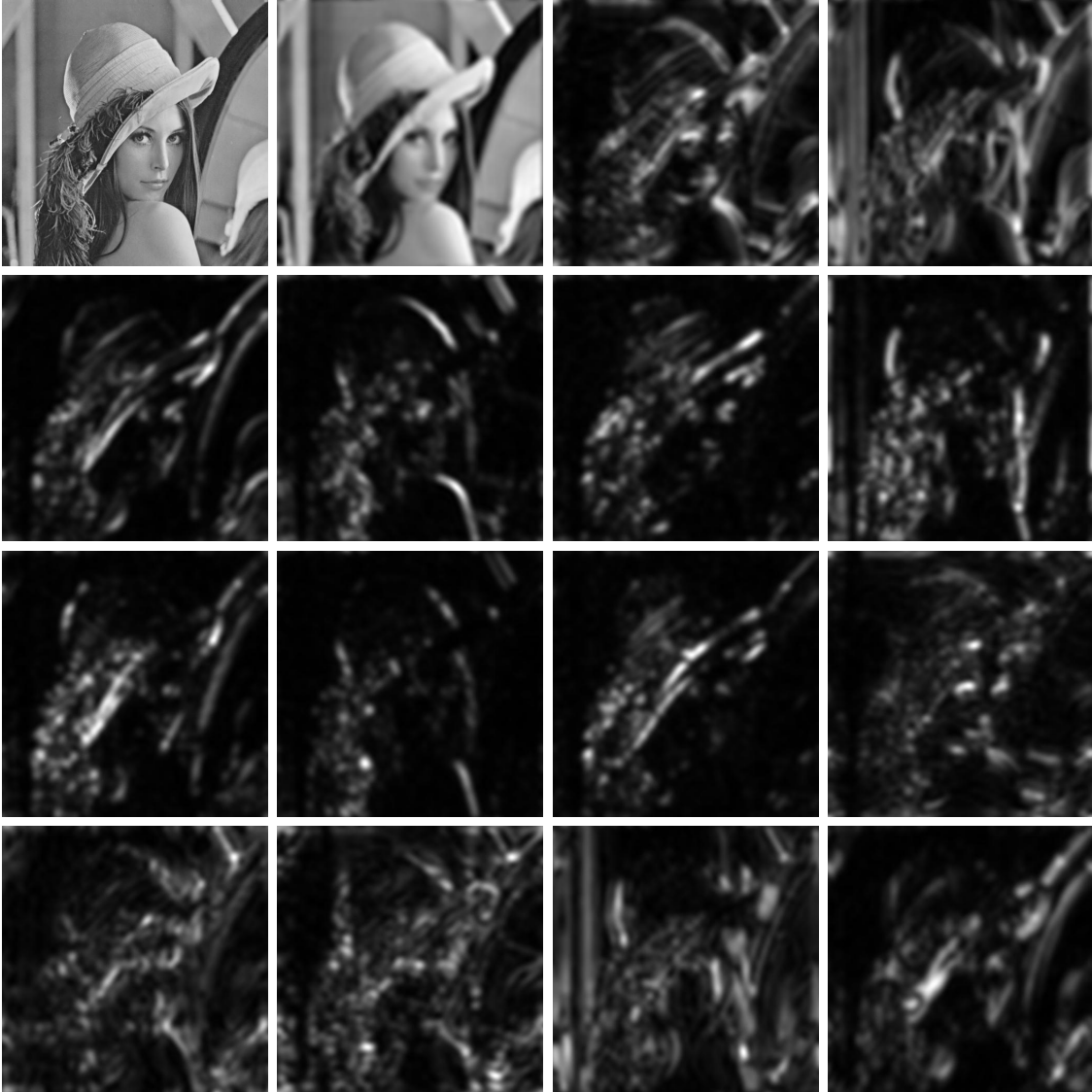


FIGURE 5.1. The first image is the original Lena image. The second image is the zero-order coefficient $f * f_0$. The third to eleventh images are first-order coefficients, $|f * f_p| * f_0$, for

$$p = (0, 1), (1, 0), (1, 1), (1, -1), (1, 2), (2, 0), (2, 1), (2, -1), (2, 2),$$

respectively. The last five images correspond to the second-order coefficients, $| |f * f_p| * f_q| * f_0$, for

$$(p, q) = (0, 1, 0, 1), (0, 1, 1, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1),$$

respectively.

We used Gabor frames as the model example, but our theory applies to any uniform covering frame. Additionally, our theory is flexible because the uniform

covering frame assumptions do not require the frame elements to be related to each other in a specific way, unlike wavelets.

- (b) *Upper and lower bounds.* We showed that $\mathcal{S}_{\mathcal{F}}$ is energy preserving and an appropriate truncation of $\mathcal{S}_{\mathcal{F}}$ is bounded above and below. Under the restrictive admissibility condition, Mallat showed that $\mathcal{S}_{\mathcal{W}}$ is also energy preserving. However, Mallat's argument cannot be extended to the case where $\mathcal{S}_{\mathcal{W}}$ is truncated; this is because there is no energy decay estimate for the wavelet case, unlike for uniform covering frames.

In general, the generalized feature extractor Φ is not energy preserving and possibly has trivial kernel. This is not surprising, because the lower bounds on $\|\mathcal{S}_{\mathcal{F}}(f)\|_{L^2\ell^2}$ and $\|\mathcal{S}_{\mathcal{W}}(f)\|_{L^2\ell^2}$ are related to the amount of energy the complex modulus pushes from high to low frequencies from one layer to the next. Thus, it would be surprising if any kind of frame and any type of nonlinearity has the same kind of effect.

- (c) *Non-expansiveness.* The non-expansiveness property holds for $\mathcal{S}_{\mathcal{F}}$, $\mathcal{S}_{\mathcal{W}}$, and Φ because this is a consequence of the frame property and network structure.
- (d) *Translation contraction estimate.* Wiatowski and Bölcskei did not provide a translation estimate for Φ . Our translation estimate, Theorem 3.6c, is similar to Mallat's translation estimate [Mal12, Theorem 2.10],

$$\|\mathcal{S}_{\mathcal{W}}(T_y f) - \mathcal{S}_{\mathcal{W}}(f)\|_{L^2\ell^2} \leq C2^{-J}|y| \left(\sum_{k=0}^{\infty} \sum_{\lambda \in \Lambda^k} \|U[\lambda]f\|_{L^2}^2 \right).$$

Indeed, our $\|\nabla f_0\|_{L^1}$ plays the same role as his $C2^{-J}$ because if $f_0(x) = 2^{-dJ}\phi(2^{-J}x)$ for some smooth $\phi \in L^1(\mathbb{R}^d)$, like in Mallat's case, then $\|\nabla f_0\|_{L^1} = \|\nabla \phi\|_{L^1}2^{-J} = C2^{-J}$. The only difference is that our inequality is more transparent because it depends on $\|f\|_{L^2}$, whereas Mallat's estimate depends on the more complicated term $\sum_{k=0}^{\infty} \sum_{\lambda \in \Lambda^k} \|U[\lambda]f\|_{L^2}^2$. This term is finite if f belongs to a certain logarithmic Sobolev space and ψ satisfies the admissibility condition.

- (e) *Diffeomorphism contraction estimate.* Our diffeomorphism estimate, Theorem 3.6d, is essentially identical to [WB15, Theorem 1]. In contrast, Mallat's diffeomorphism estimate [Mal12, Theorem 2.12],

$$\|\mathcal{S}_{\mathcal{W}}(T_{\tau} f) - \mathcal{S}_{\mathcal{W}}(f)\|_{L^2\ell^2} \leq C(J, \tau) \sum_{k=0}^{\infty} \sum_{\lambda \in \Lambda^k} \|U[\lambda]f\|_{L^2}$$

is quite different. At a first glance, it appears that his theorem requires much weaker assumptions on f since ours requires the assumption that f is almost band-limited. However, Mallat's estimate is only meaningful if $\sum_{k=0}^{\infty} \sum_{\lambda \in \Lambda^k} \|U[\lambda]f\|_{L^2}^2 < \infty$. It is unclear what functions satisfy this condition, and we expect that characterizing this class of functions is a difficult task.

- (f) *Rotational invariance.* By exploiting that \mathcal{W} is partially generated by a finite rotation group G , Mallat defined a variant of the windowed scattering operator

$\tilde{\mathcal{S}}_{\mathcal{W}}$ that is G -invariant: $\tilde{\mathcal{S}}_{\mathcal{W}}(f \circ r) = \tilde{\mathcal{S}}_{\mathcal{W}}(f)$ for all $f \in L^2(\mathbb{R}^d)$ and $r \in G$, see [Mal12, Section 5]. In the companion paper [CL16], we construct uniform covering frames that are partially generated by G , and define the G -invariant rotational Fourier scattering transform $\tilde{\mathcal{S}}_{\mathcal{F}}$. In general, it is not possible to adapt Φ and obtain a G -invariant $\tilde{\Phi}$, because the underlying frame elements need to be “compatible” with the action of G .

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